ON VERIFYING MATHEMATICAL MODELS WITH DIFFUSION, TRANSPORT AND INTERACTION

Messoud Efendiev, Stefanie Sonner

Abstract

We present a criterion for preserving the positive cone for a large class of quasi-linear parabolic systems. It is easy to apply and allows to verify mathematical models. As an application we consider new models arising in the modelling of biofilms.

Keywords: Necessary and Sufficient Conditions, Positive Cone, Quasi-linear Parabolic Systems, Biofilm Model, Chemotaxis, Quorum Sensing

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1 Introduction

The solutions of many systems of convection-diffusion-reaction equations arising in biology, physics or engineering describe such quantities as population densities, pressure or concentrations of nutrients and chemicals. Thus, a natural property to require for the solutions is positivity. Models that do not guarantee positivity are not valid or break down for small values of the solution. In many cases, showing that a particular model does not preserve positivity leads to a better understanding of the model and its limitations. One of the first steps in analyzing ecological or biological models mathematically is to test whether solutions originating from non-negative initial data remain non-negative (as long as they exist). In other words, the model under consideration ensures that the positive cone is positively invariant.

For scalar equations the non-negativity of solutions emanating from non-negative initial data is a direct consequence of the maximum principle, see [13] and [11].

However, for systems of equations the maximum principle is not valid. In the particular case of monotone systems the situation resembles the case of scalar equations, sufficient conditions for preserving the positive cone can be found in [14]. In this article we will formulate a criterion, that is, we will state necessary and sufficient conditions, for the positivity of solutions of systems of quasi-linear and semi-linear convection-diffusion-reaction-equations. It provides the modeler with a tool, which is easy to verify, to approach the question of positive invariance of the model. It turns out that for semi-linear systems, the diffusion and convection matrices need to be diagonal, while the quasi-linear case is essentially different. Here, cross-diffusion and -convection terms are allowed, however, the matrices are of a very particular form.

Our concern is not the existence but the qualitative behavior of the solutions. In order to formulate our criterion we will assume that for arbitrary initial data a unique solution of the boundary-value problem exists. As a consequence of our main theorem we derive comparison theorems for systems of quasi-linear and semi-linear convection-diffusion-reaction equations.

2 Main Result

We consider systems of quasi-linear equations of the form

$$\begin{cases} \partial_t u = a(u) \cdot \Delta u - \gamma(u) \cdot Du + f(u) & \Omega \times (0,T] \\ u|_{t=0} = u_0 & \Omega \times \{0\} \\ u|_{\partial\Omega} = 0 & \partial\Omega \times [0,T], \end{cases} \tag{2.1}$$

where u is a vector-valued function $u(x,t) = (u^1(x,t), \ldots, u^k(x,t))$ of $x \in \Omega$ and $t \in [0,T]$. We assume $T > 0$ and Ω is an open and bounded subset of $\mathbb{R}^n$, $n \in \mathbb{N}$.

Assumption 2.1. *The diffusion matrix* $a = (a_{ij}(u))_{1\le i,j,\le k}$ *is a* $k \times k$*-matrix with density-dependent coefficient functions* $a_{ij} : \mathbb{R}^k \to \mathbb{R}$ *and satisfies the parabolicity assumption* $a(u) + a(u)^* > 0$*. Moreover, let the interaction term* $f(u) = (f_1(u), \ldots, f_k(u))$ *satisfy* $f \in C^1(\mathbb{R}^k; \mathbb{R}^k)$ *and the diffusion term be given by* $\gamma(u) \cdot \mathrm{D}u := \sum_{l=1}^n \gamma^l(u) \cdot \partial_{x_l} u$*, where the coefficient functions of the* $k \times k$*-matrices* $\gamma^l(u) = (\gamma^l_{ij}(u))_{1\le i,j\le k}$ *depend on* u*. The derivatives* ∂_{x_l}*, as well as the Laplacian* $\Delta = \Delta_x$*, are applied componentwise.*

In order to formulate our criterion for the positivity of solutions of this system of parabolic equations, we define the positive cone as the set of non-negative vector-valued functions with components in $L^2(\Omega)$. Our aim is to prove that the positive cone is a positively invariant region of system (2.1).

Definition 2.2. *By* $K^+ := \{u : \Omega \to \mathbb{R}^k \mid u^i \in L^2(\Omega),\ u^i \geq 0 \ a.e.,\ i = 1, \dots, k\}$ *we denote the* ***positive cone****, that is the set of all non-negative vector-valued functions on* Ω.

We especially emphasize that in the sequel we assume that for any initial data $\mathbf{u_0} \in \mathbf{K}^+$ **there exists a unique solution of system (2.1) and for all t the solution and their derivatives with respect to x satisfy** $\mathbf{L}^\infty$**-estimates.** This property of the solutions will be essential for the proof of our main theorem. Sufficient conditions on the data justifying this assumption can be found in [10], [4] and [5].

The following theorem provides a criterion, which ensures that all solutions $u(\,\cdot\,,\,\cdot\,;u_o) : \Omega \times [0,T] \to \mathbb{R}^k$ of system (2.1) originating from non-negative initial data $u_0 \in K^+$ remain non-negative (as long as they exist).

Theorem 2.3. *Let* f *and the matrices* a, γ^l, $1 \leq l \leq n$, *fulfill the assumptions in 2.1. Moreover, we assume the coefficient functions of* a *and* γ *are in* $C^1(\mathbb{R}^k;\mathbb{R})$, *the second partial derivatives of* a_{ij} *for* $1 \leq i,j \leq k, i \neq j$, *exist and belong to the space* $L^\infty_{loc}(\mathbb{R}^k)$. *Moreover, let the initial data* $u_0 \in K^+$ *and satisfy* $u_0|_{\partial\Omega} = 0$. *Then, the solution remains non-negative, that is* $u(\,.\,,t;u_0) \in K^+$ *for* $t > 0$, *if and only if the interaction term satisfies*

$$f_i(u^1, \dots, \underbrace{0}_{i}, \dots, u^k) \geq 0 \quad \textit{for } u^1 \geq 0, \dots, u^k \geq 0$$

and for the matrices a *and* γ^l

$$a_{ij}(u^1, \dots, \underbrace{0}_{i}, \dots, u^k) = \gamma^l_{ij}(u^1, \dots, \underbrace{0}_{i}, \dots, u^k) = 0$$

holds for all $1 \leq i,j \leq k$, $i \neq j$, *and* $1 \leq l \leq n$. *This implies, the matrices can be represented as*

$$a(u) = \begin{pmatrix} a_{11}(u) & u^1 \cdot A_{12}(u) & u^1 \cdot A_{13}(u) & \cdots & u^1 \cdot A_{1k}(u) \\ u^2 \cdot A_{21}(u) & a_{22}(u) & u^2 \cdot A_{23}(u) & \cdots & u^2 \cdot A_{2k}(u) \\ \vdots & \vdots & \vdots & & \vdots \\ u^k \cdot A_{k1}(u) & u^k \cdot A_{k2}(u) & u^k \cdot A_{k3}(u) & \cdots & a_{kk}(u) \end{pmatrix}$$

$$\gamma^l(u) = \begin{pmatrix} \gamma^l_{11}(u) & u^1 \cdot \Gamma^l_{12}(u) & u^1 \cdot \Gamma^l_{13}(u) & \cdots & u^1 \cdot \Gamma^l_{1k}(u) \\ u^2 \cdot \Gamma^l_{21}(u) & \gamma_{22}(u) & u^2 \cdot \Gamma^l_{23}(u) & \cdots & u^2 \cdot \Gamma^l_{2k}(u) \\ \vdots & \vdots & \vdots & & \vdots \\ u^k \cdot \Gamma^l_{k1}(u) & u^k \cdot \Gamma^l_{k2}(u) & u^k \cdot \Gamma^l_{k3}(u) & \cdots & \gamma_{kk}(u) \end{pmatrix}$$

with bounded functions $A_{ij}(u)$ and $\Gamma^l_{ij}(u)$, $i \neq j, 1 \leq l \leq n$.

Proof. ***Necessity:*** We assume the solution $u(\,\cdot\,, t; u_0)$ corresponding to initial data $u_0 \in K^+$ remains non-negative for $t > 0$ and prove the necessity of the stated conditions. In the following we will make formal calculations, for its validity we refer to [10]. Taking smooth initial data u_0 and an arbitrary function $v \in K^+$, that is orthogonal to u_0, we obtain

$$\begin{aligned}
\big(\partial_t u|_{t=0}, v\big)_{L^2(\Omega;\mathbb{R}^k)} &= \big(\lim_{t\to 0_+} \frac{u(\,\cdot\,, t; u_0) - u_0}{t}, v\big)_{L^2(\Omega;\mathbb{R}^k)} = \\
&= \lim_{t\to 0_+} \big(\frac{u(\,\cdot\,, t; u_0)}{t}, v\big)_{L^2(\Omega;\mathbb{R}^k)} - \lim_{t\to 0_+} \big(\frac{u_0}{t}, v\big)_{L^2(\Omega;\mathbb{R}^k)} = \\
&= \lim_{t\to 0_+} \big(\frac{u(\,\cdot\,, t; u_0)}{t}, v\big)_{L^2(\Omega;\mathbb{R}^k)} \geq 0,
\end{aligned}$$

where we used the orthogonality of u_0 and v as well as the assumption $u(\,\cdot\,, t; u_0) \in K^+$. On the other hand, since u is the solution of system (2.1) corresponding to initial data u_0, we obtain

$$\big(\partial_t u|_{t=0}, v\big)_{L^2(\Omega;\mathbb{R}^k)} = \big(a(u_0)\cdot \Delta u_0 - \gamma(u_0)\cdot \mathrm{D}u_0 + f(u_0), v\big)_{L^2(\Omega;\mathbb{R}^k)} \geq 0. \qquad (2.2)$$

In particular, for fixed $i \in \{1,\dots,k\}$ choosing the functions $u_0 = (\tilde{u}^1, \dots, \underbrace{0}_{i}, \dots, \tilde{u}^k)$ and $v = (0, \dots, \underbrace{\tilde{v}}_{i}, \dots, 0)$ with $\tilde{u}^1 \geq 0, \dots, \tilde{u}^k \geq 0$, $\tilde{v} \geq 0$, leads to the scalar inequality

$$\int_\Omega \Big(\sum_{j=1, j\neq i}^{k} a_{ij}(u_0)\cdot \Delta \tilde{u}^j - \sum_{l=1}^{n} \sum_{j=1, j\neq i}^{k} \gamma^l_{ij}(u_0)\cdot \partial_{x_l}\tilde{u}^j + f_i(u_0)\Big)\cdot \tilde{v}\, dx \geq 0.$$

As this inequality holds for arbitrary non-negative $\tilde{v} \in L^2(\Omega)$, we get the pointwise estimate

$$\sum_{j=1, j\neq i}^{k} a_{ij}(u_0)\cdot \Delta \tilde{u}^j - \sum_{l=1}^{n} \sum_{j=1, j\neq i}^{k} \gamma^l_{ij}(u_0)\cdot \partial_{x_l}\tilde{u}^j + f_i(u_0) \geq 0 \quad a.e. \text{ in } \Omega. \qquad (2.3)$$

This implies for $1 \leq j \leq k$, $j \neq i$ and all $1 \leq l \leq n$

$$a_{ij}(\tilde{u}^1, \dots, 0, \dots, \tilde{u}^k) = \gamma^l_{ij}(\tilde{u}^1, \dots, 0, \dots, \tilde{u}^k) = 0.$$

Moreover, $f_i(\tilde{u}^1, \dots, 0, \dots, \tilde{u}^k) \geq 0$ for $\tilde{u}^1 \geq 0, \dots, \tilde{u}^k \geq 0$. Hence, the matrices a and γ^l are necessarily of the stated form and the components of the interaction term f_i, $1 \le i \le k$, satisfy $f_i(u^1, \dots, \underbrace{0}_{i}, \dots, u^k) \ge 0$ for $u^1 \ge 0, \dots, u^k \ge 0$.

Sufficiency: We show that under the given conditions on a, γ and f the solution corresponding to initial data $u_0 \in K^+$ remains non-negative for $t > 0$. In this case, the system of equations takes the form

$$\partial_t u^i = a_{ii}(u) \cdot \Delta u^i + \sum_{j=1, j\neq i}^{k} u^i \cdot A_{ij}(u) \cdot \Delta u^j - \sum_{l=1}^{n} \gamma_{ii}^l(u) \cdot \partial_{x_l} u^i - \\ - \sum_{l=1}^{n} \sum_{j=1, j\neq i}^{k} u^i \cdot \Gamma_{ij}^l(u) \cdot \partial_{x_l} u^j + f_i(u),$$

$1 \le i \le k$, where the functions $A_{ij}, \Gamma_{ij}^l : \mathbb{R}^k \to \mathbb{R}$ are defined as

$$A_{ij}(u) := \int_0^1 \partial_i a_{ij}(u^1, \dots, su^i, \dots, u^k) ds,$$
$$\Gamma_{ij}^l(u) := \int_0^1 \partial_i \gamma_{ij}^l(u^1, \dots, su^i, \dots, u^k) ds.$$

Let $u \in L^2(\Omega)$. Introducing its positive and negative part $u_+ := \max\{u, 0\}$, respectively $u_- := \max\{-u, 0\}$, we can represent $u = u_+ - u_-$ and $|u| = u_+ + u_-$. By the definition immediately follows $u_-\, u_+ = 0$. It is a well-known fact that for $u \in \mathrm{H}^1(\Omega)$ also $u_+, u_- \in \mathrm{H}^1(\Omega)$ holds and

$$Du_- = \begin{cases} -Du & u < 0 \\ 0 & u \ge 0 \end{cases} \qquad Du_+ = \begin{cases} Du & u > 0 \\ 0 & u \le 0 \end{cases}$$

(cf. [8]). This certainly implies

$$Du_+\, u_- = u_+\, Du_- = Du_+\, Du_- = 0.$$

In order to prove the positivity of the solution $u = u(\,\cdot\,, \cdot\,; u_o)$ corresponding to initial data $u_0 \in K^+$ we show that $(u_0^i)_- = 0$ a.e. implies $u_-^i := u^i(\,\cdot\,, t; u_o)_- = 0$ a.e. for $t > 0$, $1 \le i \le k$. Multiplying the i-th equation by u_-^i and integrating over Ω yields

$$\begin{aligned} &(\partial_t u^i, u_-^i)_{L^2(\Omega)} \\ &= (a_{ii}(u) \cdot \Delta u^i, u_-^i)_{L^2(\Omega)} + \sum_{j=1, j\neq i}^{k} (u^i \cdot A_{ij}(u) \cdot \Delta u^j, u_-^i)_{L^2(\Omega)} - \\ &- \sum_{l=1}^{n} (\gamma_{ii}^l(u) \cdot \partial_{x_l} u^i, u_-^i)_{L^2(\Omega)} - \sum_{l=1}^{n} \sum_{j=1, j\neq i}^{k} (u^i \cdot \Gamma_{ij}^l(u) \cdot \partial_{x_l} u^j, u_-^i)_{L^2(\Omega)} + \\ &+ (f_i(u), u_-^i)_{L^2(\Omega)}. \end{aligned}$$

Note that the left-hand side of the equation can be written as

$$(\partial_t u^i, u^i_-)_{L^2(\Omega)} = -(\partial_t u^i_-, u^i_-)_{L^2(\Omega)} = -\frac{1}{2}\partial_t \|u^i_-\|^2_{L^2(\Omega)}.$$

By assumption the solution satisfies $u^i \in L^\infty(\Omega)$ as well as $\partial_{x_j} u^i \in L^\infty(\Omega)$, for all $1 \le i \le k$ and $1 \le j \le n$. Taking into account the homogeneous Dirichlet boundary conditions we obtain for the first term of the right-hand side of the equation

$$\begin{aligned}
&(a_{ii}(u) \cdot \Delta u^i, u^i_-)_{L^2(\Omega)} \\
&\quad = -(a_{ii}(u) \cdot \Delta u^i_-, u^i_-)_{L^2(\Omega)} = \int_\Omega \nabla(a_{ii}(u) \cdot u^i_-) \cdot \nabla u^i_- dx = \\
&\quad = \int_\Omega a_{ii}(u) \cdot |\nabla u^i_-|^2 dx + \sum_{j=1}^k \int_\Omega \partial_j a_{ii}(u) \cdot \nabla u^j \cdot u^i_- \cdot \nabla u^i_- dx.
\end{aligned}$$

Furthermore, the second integral can be estimated by

$$\sum_{j=1}^k \int_\Omega |\partial_j a_{ii}(u) \cdot \nabla u^j \cdot u^i_- \cdot \nabla u^i_-| dx \le C_1 \cdot \int_\Omega \sum_{m=1}^n |\partial_{x_m} u^i_-| \cdot |u^i_-| dx,$$

for some constant $C_1 \ge 0$. Here, we used the boundedness of the functions u^j and their derivatives $\partial_{x_m} u^j$, $1 \le m \le n$, as well as the assumption $a_{ii} \in C^1(\mathbb{R}^k)$. For the second diffusion term we obtain

$$\begin{aligned}
&\big|\Big(\sum_{j=1, j\neq i}^k u^i \cdot A_{ij}(u) \cdot \Delta u^j, u^i_-\Big)_{L^2(\Omega)}\big| = \big| -\sum_j \int_\Omega A_{ij}(u) \cdot (u^i_-)^2 \cdot \Delta u^j dx\big| = \\
&= \big|\sum_j \int_\Omega \nabla(A_{ij}(u) \cdot (u^i_-)^2) \cdot \nabla u^j dx\big| \le \\
&\le \sum_j \Big(\int_\Omega 2 \cdot |A_{ij}(u) \cdot u^i_- \cdot \nabla u^i_- \cdot \nabla u^j| dx + \\
&+ \int_\Omega \sum_{l=1}^k |\partial_l A_{ij}(u) \cdot \nabla u^l \cdot (u^i_-)^2 \cdot \nabla u^j| dx\Big) \le \\
&\le C_2 \cdot \int_\Omega \sum_{m=1}^n |\partial_{x_m} u^i_-| \cdot |u^i_-| dx + C_3 \cdot \|u^i_-\|^2_{L^2(\Omega)},
\end{aligned}$$

for some constants $C_2, C_3 \ge 0$. As before, we used the fact that u^j and $\partial_{x_m} u^j$ are in $L^\infty(\Omega)$, for all $1 \le j \le k, 1 \le m \le n$, and the assumption $\partial_l \partial_i a_{ij} \in L^\infty_{loc}(\Omega)$.

Similarly, we derive an estimate for the convection terms

$$\begin{aligned}
&\Big| -\sum_{l=1}^{n}(\gamma_{ii}^{l}(u)\cdot\partial_{x_l}u^{i},u_{-}^{i})_{L^2(\Omega)} - \sum_{l=1}^{n}\sum_{j=1,j\neq i}^{k}(u^{i}\cdot\Gamma_{ij}^{l}(u)\cdot\partial_{x_l}u^{j},u_{-}^{i})_{L^2(\Omega)}\Big| \leq \\
&\leq \int_{\Omega}\sum_{l=1}^{n}\big|\gamma_{ii}^{l}(u)\cdot\partial_{x_l}u_{-}^{i}\cdot u_{-}^{i}\big|dx + \int_{\Omega}\sum_{l=1}^{n}\sum_{j=1,j\neq i}^{k}\big|\Gamma_{ij}^{l}(u)\cdot\partial_{x_l}u^{j}\cdot(u_{-}^{i})^2\big|dx \leq \\
&\leq C_4\cdot\int_{\Omega}\sum_{m=1}^{n}|\partial_{x_m}u_{-}^{i}|\cdot|u_{-}^{i}|dx + C_5\cdot\|u_{-}^{i}\|_{L^2(\Omega)}^2,
\end{aligned}$$

for some constants $C_4, C_5 \geq 0$. Here, we used the assumption $\gamma_{ij}^{l} \in C^1(\mathbb{R}^k)$ and the boundedness of u^j and $\partial_{x_m}u^j$, $1 \leq j \leq k, 1 \leq m \leq n$. It remains to estimate the interaction term. By assumption $f \in C^1(\mathbb{R}^k;\mathbb{R}^k)$, so we can write

$$f_i(u^1,\dots,u^k) = f_i(u^1,\dots,\underbrace{0}_{i},\dots,u^k) + u^i\cdot\int_0^1 \partial_i f_i(u^1,\dots,su^i,\dots,u^k)ds,$$

that is, $f_i(u^1,\dots,u^k) = f_i(u^1,\dots,0,\dots,u^k) + u^i\cdot F_i(u^1,\dots,u^k)$, with a bounded function $F_i : \mathbb{R}^k \to \mathbb{R}$. Consequently, the last integral yields

$$\begin{aligned}
\int_{\Omega} f_i(u)\cdot u_{-}^{i}dx &= \int_{\Omega} f_i(u^1,\dots,\underbrace{0}_{i},\dots,u^k)\cdot u_{-}^{i} + u^i\cdot F_i(u^1,\dots,u^k)\cdot u_{-}^{i}dx = \\
&= \int_{\Omega} f_i(u^1,\dots,\underbrace{0}_{i},\dots,u^k)\cdot u_{-}^{i}dx - \int_{\Omega}|u_{-}^{i}|^2\cdot F_i(u^1,\dots,u^k)dx.
\end{aligned}$$

Summing up all terms we obtain

$$\begin{aligned}
&\frac{1}{2}\partial_t\|u_{-}^{i}\|_{L^2(\Omega)}^2 + \int_{\Omega}a_{ii}(u)\cdot|\nabla u_{-}^{i}|^2dx \leq \\
&\leq C_6\cdot\int_{\Omega}\sum_{m=1}^{n}|\partial_{x_m}u_{-}^{i}|\cdot|u_{-}^{i}|dx + C_7\cdot\|u_{-}^{i}\|_{L^2(\Omega)}^2 - \\
&-\int_{\Omega} f_i(u^1,\dots,\underbrace{0}_{i},\dots,u^k)\cdot u_{-}^{i}dx,
\end{aligned}$$

for some constants $C_6, C_7 \geq 0$. Taking into account the hypothesis $a_{ii}(u) > 0$ and using Young's inequality to estimate the terms

$$\int_{\Omega}\sum_{m=1}^{n}|\partial_{x_m}u_{-}^{i}|\cdot|u_{-}^{i}|dx \leq \int_{\Omega}\epsilon|\nabla u_{-}^{i}|^2 + C_\epsilon|u_{-}^{i}|^2dx$$

follows

$$\frac{1}{2}\partial_t \|u^i_-\|^2_{L^2(\Omega)} \le c \cdot \|u^i_-\|^2_{L^2(\Omega)} - \int_\Omega f_i(u^1, \dots, \underbrace{0}_{i}, \dots, u^k) \cdot u^i_- dx,$$

for some constant $c \ge 0$, if we choose $\epsilon > 0$ sufficiently small. Under the hypothesis $f_i(u^1, \dots, 0, \dots, u^k) \ge 0$ (which was a priori only assumed for $u^1 \ge 0, \dots u^k \ge 0$) the inequality

$$\partial_t \|u^i_-\|^2_{L^2(\Omega)} \le c \cdot \|u^i_-\|^2_{L^2(\Omega)},$$

follows. By Gronwall's Lemma and the initial condition $(u^i_0)_- = 0$ we conclude $\|u^i_-\|_{L^2(\Omega)} = 0$, that is $u^i_- = 0$ a.e. in Ω.

It remains to justify our assumptions on the functions f_i. Instead of the original system (2.1) we consider the modified system

$$\begin{cases} \partial_t \hat{u} = a(\hat{u}) \cdot \Delta \hat{u} - \gamma(\hat{u}) \cdot D\hat{u} + \hat{f}(\hat{u}) & \Omega \times (0, T] \\ \hat{u}|_{t=0} = u_0 & \Omega \times \{0\} \\ \hat{u}|_{\partial\Omega} = 0 & \partial\Omega \times [0, T], \end{cases}$$

where the function $\hat{f}$ is given by

$$\hat{f}_i(\hat{u}^1, \dots, \hat{u}^k) = f_i(|\hat{u}^1|, \dots, 0, \dots, |\hat{u}^k|) + u^i \cdot F_i(\hat{u}^1, \dots, \hat{u}^k)$$

with F_i as defined above. Following the same arguments we conclude for the solution $\hat{u}$ of this modified system that $u_0 \in K^+$ implies $\hat{u}(\,\cdot\,, t; u_0) \in K^+$ for $t > 0$. However, this solution $\hat{u}$ with $\hat{u}^1 \ge 0, \dots, \hat{u}^k \ge 0$ satisfies the original system

$$\begin{cases} \partial_t u = a(u) \cdot \Delta u - \gamma(u) \cdot Du + f(u) & \Omega \times (0, T] \\ u|_{t=0} = u_0 & \Omega \times \{0\} \\ u|_{\partial\Omega} = 0 & \partial\Omega \times [0, T], \end{cases}$$

By the uniqueness of the solution corresponding to initial data u_0 follows $u = \hat{u}$, which implies $u(\,\cdot\,, t; u_0) \in K^+$ for $t > 0$, and concludes the proof of the theorem. □

3 Semi-linear Case

A special case of the systems considered in Section 2 are systems of semi-linear convection-diffusion-reaction equations of the form

$$\begin{cases} \partial_t u = a \cdot \Delta u - \gamma \cdot Du + f(u) & \Omega \times (0, T] \\ u|_{t=0} = u_0 & \Omega \times \{0\} \\ u|_{\partial\Omega} = 0 & \partial\Omega \times [0, T], \end{cases} \tag{3.4}$$

where $u(x,t) = (u^1(x,t), \dots, u^k(x,t))$ is a vector-valued function of $x \in \Omega,\ t \in [0,T]$, $T > 0$ and $\Omega \subset \mathbb{R}^n$ is open and bounded.

Assumption 3.1. *The diffusion matrix $a = (a_{ij})_{1\le i,j,\le k}$ is a $k \times k$-matrix with constant coefficients $a_{ij} \in \mathbb{R}$ and satisfies $a + a^* > 0$. Moreover, let $f \in C^1(\mathbb{R}^k; \mathbb{R}^k)$, $f(u) = (f_1(u), \dots, f_k(u))$, and the diffusion term be given by $\gamma \cdot \mathrm{D}u := \sum_{l=1}^n \gamma^l \cdot \partial_{x_l} u$, where the matrices $\gamma^l = (\gamma^l_{ij})_{1\le i,j\le k}$ have constant coefficients $\gamma^l_{ij} \in \mathbb{R}$.*

In this section we assume that for any initial data $\mathbf{u_0} \in \mathbf{K}^+$ there exists a unique solution of system (3.4) and for all t the solution satisfies $\mathbf{L}^\infty$-estimates. Note that we do not assume the boundedness of the derivatives of the solution as in Section 2. The following theorem shows that, contrary to the quasi-linear case, system (3.4) preserves the positivity of solutions only if the diffusion and convection matrices are diagonal.

Theorem 3.2. *Let the matrices a and $\gamma^l, 1 \le l \le n$, and the interaction term f fulfill the assumptions in 3.1. Moreover, we assume the initial data $u_0 \in K^+$ satisfies $u_0|_{\partial\Omega} = 0$. Then, necessary and sufficient conditions for the positivity of the solution are the following: The matrices a and γ are diagonal and the components of the reaction term satisfy*

$$f_i(u^1, \dots, \underbrace{0}_{i}, \dots, u^k) \ge 0, \quad \text{for } u^1 \ge 0, \dots, u^k \ge 0,$$

and all $1 \le i \le k$.

Proof. ***Necessity:*** For the convenience of the reader we will point out where the proof simplifies in comparison to the proof of Theorem 2.3. We follow the same arguments to deduce the inequality

$$\big(\partial_t u|_{t=0}, v\big)_{L^2(\Omega;\mathbb{R}^k)} = \big(a \cdot \Delta u_0 - \gamma \cdot \mathrm{D}u_0 + f(u_0), v\big)_{L^2(\Omega;\mathbb{R}^k)} \ge 0 \tag{3.5}$$

for an arbitrary function $v \in K^+$, which is orthogonal to u_0. Choosing the functions $u_0 = (0, \dots, \underbrace{\tilde u}_{i}, \dots, 0)$ and $v = (0, \dots, \underbrace{\tilde v}_{j}, \dots, 0)$ with $\tilde u \ge 0,\ \tilde v \ge 0$ and $i \ne j$, leads to the pointwise estimate

$$a_{ji} \cdot \Delta \tilde u - \sum_{l=1}^n \gamma^l_{ji} \cdot \partial_{x_l} \tilde u + f_j(0 \dots, \underbrace{\tilde u}_{i}, \dots, 0) \ge 0 \quad a.e. \text{ in } \Omega. \tag{3.6}$$

We prove by contradiction that the diffusion- and convection-matrices are necessarily diagonal. Let us assume $a_{ji} \ne 0$. We may choose a function that attains its maximum in $x_0 \in \Omega$ and whose second derivative is arbitrarily negative, for instance, $\tilde{u}(x) := e^{-\frac{1}{\epsilon}(x^1-x_0^1)^2}$. We compute $\partial_{x_1}\tilde{u}(x) = -\frac{2}{\epsilon}(x^1-x_0^1)\cdot e^{-\frac{1}{\epsilon}(x^1-x_0^1)^2}$ and $\Delta\tilde{u}(x) = -\frac{2}{\epsilon}e^{-\frac{1}{\epsilon}(x^1-x_0^1)^2} + \frac{4}{\epsilon^2}(x^1-x_0^1)^2\cdot e^{-\frac{1}{\epsilon}(x^1-x_0^1)^2}$, hence, $\partial_{x_l}\tilde{u}(x_0) = 0$, $1 \le l \le n$, and $\Delta\tilde{u}(x_0) = -\frac{2}{\epsilon}$. As ϵ can be chosen arbitrarily small, it contradicts inequality (3.6) in the point x_0. Next, we choose the function $\tilde{\tilde{u}}(x) := e^{-\frac{1}{\epsilon}(x^1-x_0^1)}$ to derive $\gamma_{ji}^l = 0$, $1 \le l \le n$.

This implies the matrices a and γ^l, $1 \le l \le n$, are diagonal. Finally, we use the functions $u_0 = (\tilde{u}^1, \ldots, \underbrace{0}_{i}, \ldots, \tilde{u}^k)$ and $v = (0, \ldots, \underbrace{\tilde{v}}_{i}, \ldots, 0)$, with $\tilde{u}^1 \ge 0, \ldots, \tilde{u}^k \ge 0$, $\tilde{v} \ge 0$ and inequality (3.6) to conclude $f_i(\tilde{u}^1, \ldots, \underbrace{0}_{i}, \ldots, \tilde{u}^k) \ge 0$ for $\tilde{u}^1 \ge 0, \ldots, \tilde{u}^k \ge 0$.

***Sufficiency*:** We assume the matrices a and γ are diagonal, let $a = diag(a_1, \ldots, a_k)$ and $\gamma^l = diag(\gamma_1^l, \ldots, \gamma_k^l)$, and f satisfies the assumptions of the theorem. The system takes the form

$$\partial_t u^i = a_i \cdot \Delta u^i - \sum_{l=1}^{n} \gamma_i^l \cdot \partial_{x_l} u^i + f_i(u^1, \ldots, u^k), \quad i = 1, \ldots, k.$$

Multiplying the i-th equation by u_-^i and integrating over Ω yields

$$-\frac{1}{2}\partial_t\|u_-^i\|^2 \;=\; a_i\int_\Omega |\nabla u_-^i|^2 dx + \sum_{l=1}^{n}\gamma_i^l\int_\Omega \partial_{x_l}u_-^i\cdot u_-^i dx + \int_\Omega f_i(u^1,\ldots,u^k)\cdot u_-^i dx$$

First, we estimate the convection term using Young's inequality

$$|\sum_{l=1}^{n}\gamma_i^l\int_\Omega \partial_{x_l}u_-^i\cdot u_-^i dx| \;\le\; \int_\Omega \epsilon|\nabla u_-^i|^2 + C(\epsilon)|u_-^i|^2 dx,$$

for some constant $C(\epsilon) \ge 0$. The last integral is expressed exactly as in the proof of Theorem 2.3. Hence, collecting the terms we get

$$\begin{aligned}&\frac{1}{2}\partial_t\|u_-^i\|^2 + a_i\int_\Omega |\nabla u_-^i|^2 dx \le \\ &\le \int_\Omega \epsilon|\nabla u_-^i|^2 + C(\epsilon)|u_-^i|^2 dx - \int_\Omega f_i(u^1,\ldots,0,\ldots,u^k)\cdot u_-^i dx + C\|u_-^i\|^2_{L^2(\Omega)},\end{aligned}$$

for some constant $C \ge 0$. Choosing ϵ sufficiently small leads to the estimate

$$\partial_t\|u_-^i\|^2_{L^2(\Omega)} \le c\cdot\|u_-^i\|^2_{L^2(\Omega)},$$

for some constant $c \ge 0$. We used $f_i(u^1, \ldots, 0, \ldots, u^k) \ge 0$, which was a priori only assumed for $u^i \ge 0$, $1 \le i \le k$. However, the validity of this estimate is justified as in the proof of Theorem 2.3. By Gronwall's Lemma and the assumption $(u_0^i)_- = 0$ follows $u_-^i = 0$ a.e. in Ω. □

Remark 3.3. *1. Note that Theorem 2.3 can be generalized to systems of the form*

$$\partial_t u = a(u,x,t)\cdot \Delta u - \gamma(u,x,t)\cdot Du + f(u,x,t)$$

under appropriate boundedness assumptions on the coefficient functions. The (x,t)-dependence of the coefficients reflects the heterogeneity of the medium.

2. *The method used to prove that the conditions in Theorem 2.3 are sufficient for the positivity of the solutions is applicable in more general cases. As an example we consider a system of degenerate parabolic equations in Section 6.*

3. *For certain types of equations the assumption that the solution and their derivatives satisfy L^∞-estimates is not necessary. An example is the (quasi-linear) biofilm model considered in Section 7.3.*

Remark 3.4. *For semi-linear systems it is possible to deduce necessary and sufficient conditions to show that the solution originating from bounded non-negative initial data $u_0 \in [0,1]$ remains non-negative and bounded by 1. In other words,*

$$\{u \in K^+ \mid 1-u \in K^+\}$$

is an invariant region for system (3.4). To this end we consider the function $w := 1-u$ and follow the same arguments to derive conditions for the positivity of w. In addition to the conditions in Theorem 3.2 we obtain

$$f_i(u^1,\dots,1,\dots,u^k) \le 0$$

for $u^1 \le 1,\dots,u^k \le 1$. However, we need an additional assumption on the reaction term. If it satisfies $\partial_i f_i \le c$ for some constant c the claim follows.

4 Comparison Principles

Using the results of the previous sections we derive comparison theorems for systems of semi-linear and quasi-linear parabolic equations.

4.1 Semi-linear Case

Definition 4.1. *Let Ω be a subset of $\mathbb{R}^n$. We define the (partial) order relation $\preccurlyeq$ on the set of vector-valued functions on Ω. Let $u,v : \Omega \to \mathbb{R}^k$ such that $u^i, v^i \in L^2(\Omega)$. We write $u \preccurlyeq v$ if and only if $u^i \le v^i$ holds a.e. in Ω for all $1 \le i \le k$.*

Theorem 4.2. *Under the assumptions of* Theorem 3.2*, system* (3.4) *is order preserving with respect to* $\preccurlyeq$ *if and only if the matrices* a *and* γ *are diagonal and the reaction term* f *satisfies*

$$f_i(y^1, \ldots, \underbrace{y}_{i}, \ldots, y^k) \leq f_i(z^1, \ldots, \underbrace{y}_{i}, \ldots, z^k), \quad \text{whenever } y^j \leq z^j,\ j \neq i,$$

for all $1 \leq i, j \leq k$.

Proof. Let u_0 and v_0 be given initial data satisfying the assumptions of Theorem 3.2 and assume $u_0 \succcurlyeq v_0$. We prove that the order is preserved by the corresponding solutions $u = u(\,\cdot\,, t; u_0)$ and $v = v(\,\cdot\,, t; v_0)$, that is $u \succcurlyeq v$ for $t > 0$, if and only if the matrices a and γ are diagonal and the reaction term fulfills the stated monotonicity conditions. Defining the function $w := u - v$ it satisfies

$$\begin{cases} \partial_t w = a \cdot \Delta w - \gamma \cdot Dw + f(u) - f(v) & \Omega \times (0, T] \\ w|_{t=0} = w_0 & \Omega \times \{0\} \\ w|_{\partial\Omega} = 0 & \partial\Omega \times [0, T], \end{cases}$$

where $w_0 := u_0 - v_0 \in K^+$. By Theorem 3.2 follows that the solution $w = w(\,\cdot\,, t; w_0)$ is non-negative for $t > 0$ if and only if the matrices a and γ are diagonal and the function $F(w) = F(u, v) := f(u) - f(v)$ satisfies

$$F_i(w^1, \ldots, \underbrace{0}_{i}, \ldots, w^k) \geq 0 \quad \text{for } w^1 \geq 0, \ldots, w^k \geq 0$$

and all $1 \leq i \leq k$. Thus,

$$F_i(u^1 - v^1, \ldots, 0, \ldots, u^k - v^k) = f_i(u^1, \ldots, u^i, \ldots, u^k) - f_i(v^1, \ldots, u^i \ldots, v^k) \geq 0$$

for $u^j \geq v^j$, $j \neq i$, and all $1 \leq i, j \leq k$. □

Definition 4.3. *We define the more general order relation* $\gtrsim$ *on the set of vector-valued functions on* Ω*. Let* σ_1 *and* σ_2 *be disjoint and* $\sigma_1 \cup \sigma_2 = \{1, \ldots, k\}$*. For functions* u *and* v *as in Definition 4.1 we write* $u \gtrsim v$ *if and only if*

$$\begin{cases} u^j \geq v^j \ \textit{a.e.} & \text{for } j \in \sigma_1 \\ u^j \leq v^j \ \textit{a.e.} & \text{for } j \in \sigma_2. \end{cases}$$

Theorem 4.4. *Under the hypothesis of* Theorem 3.2 *system* (3.4) *is order preserving with respect to* $\precsim$ *if and only if the matrices* a *and* γ *are diagonal and the interaction term* f *satisfies*

$$\begin{aligned} f_i(y^1, \ldots, \underbrace{y}_{i}, \ldots, y^k) &\leq f_i(z^1, \ldots, \underbrace{y}_{i}, \ldots, z^k) \quad \text{if } i \in \sigma_1 \\ f_i(y^1, \ldots, \underbrace{y}_{i}, \ldots, y^k) &\geq f_i(z^1, \ldots, \underbrace{y}_{i}, \ldots, z^k) \quad \text{if } i \in \sigma_2, \end{aligned}$$

whenever

$$\begin{aligned} y^j &\le z^j && \text{if } j \in \sigma_1 \\ y^j &\ge z^j && \text{if } j \in \sigma_2 \end{aligned}$$

for $j \neq i$*.*

Proof. Let u_0 and v_0 be given initial data satisfying the assumptions of Theorem 3.2 and assume $u_0 \gtrsim v_0$. We prove that the order $\gtrsim$ is preserved by the corresponding solutions $u = u(\,\cdot\,, t; u_0)$ and $v = v(\,\cdot\,, t; v_0)$, that is $u \gtrsim v$ for $t > 0$, if and only if a and γ are diagonal and the reaction term f fulfills the stated conditions. Defining the function w by

$$w^i := \begin{cases} u^i - v^i & \text{if } i \in \sigma_1 \\ -(u^i - v^i) & \text{if } i \in \sigma_2 \end{cases}$$

it satisfies the system

$$\begin{cases} \partial_t w = \tilde{a} \cdot \mathit{\Delta} w - \tilde{\gamma} \cdot Dw + F(w) & \mathit{\Omega} \times (0,T] \\ w|_{t=0} = w_0 & \mathit{\Omega} \times \{0\} \\ w|_{\partial\mathit{\Omega}} = 0 & \partial\mathit{\Omega} \times [0,T] \end{cases}$$

with initial data $w_0 \in K^+$, where the function F is defined by

$$F_i(w) = F_i(u,v) := \begin{cases} f_i(u) - f_i(v) & \text{if } i \in \sigma_1 \\ -(f_i(u) - f_i(v)) & \text{if } i \in \sigma_2. \end{cases}$$

Furthermore, the diffusion matrix $\tilde{a}$ is given by

$$\tilde{a}_{ij} := \begin{cases} a_{ij} & \text{if } i,j \in \sigma_1 \text{ or } i,j \in \sigma_2 \\ -a_{ij} & \text{otherwise} \end{cases}$$

and the convection matrices $\tilde{\gamma}^l$ by

$$\tilde{\gamma}^l_{ij} := \begin{cases} \gamma^l_{ij} & \text{if } i,j \in \sigma_1 \text{ or } i,j \in \sigma_2 \\ -\gamma^l_{ij} & \text{otherwise} \end{cases}$$

for all $1 \le l \le n$ and $1 \le i,j \le k$. By Theorem 4.2 follows $w = w(\,\cdot\,, t; w_0) \in K^+$, that is $u \gtrsim v$ for $t > 0$, if and only if the matrices a and γ are diagonal and the function F satisfies

$$F_i(w) = \begin{cases} f_i(u^1, \dots, \underbrace{u^i}_{i}, \dots, u^k) - f_i(v^1, \dots, \underbrace{u^i}_{i}, \dots, v^k) \ge 0 & \text{for } i \in \sigma_1, \\ -\big(f_i(u^1, \dots, \underbrace{u^i}_{i}, \dots, u^k) - f_i(v^1, \dots, \underbrace{u^i}_{i}, \dots, u^k)\big) \ge 0 & \text{for } i \in \sigma_2, \end{cases}$$

whenever $w^j \ge 0$, $j \neq i$, that is $u^j \ge v^j$ if $j \in \sigma_1$ and $v^j \ge u^j$ if $j \in \sigma_2$. □

4.2 Quasi-linear Case

Next, we formulate comparison principles for the quasi-linear systems considered in Section 2.

Theorem 4.5. *In addition to the hypothesis of* Theorem 2.3 *we assume that the partial derivatives of second order of the diagonal coefficient functions a_{ii} exist and belong to the space $L^\infty_{loc}(\mathbb{R}^k)$ for all $1 \le i \le k$. Then, system* (2.1) *is order preserving with respect to $\preccurlyeq$ if and only if the matrices a and γ^l are diagonal, for the coefficient functions a_{ii} and γ_{ii}*

$$\begin{aligned} a_{ii}(u^1, \dots, \underbrace{u}_{i}, \dots, u^k) &= a_{ii}(v^1, \dots, \underbrace{u}_{i}, \dots, v^k) \\ \gamma^l_{ii}(u^1, \dots, \underbrace{u}_{i}, \dots, u^k) &= \gamma^l_{ii}(v^1, \dots, \underbrace{u}_{i}, \dots, v^k) \end{aligned}$$

holds for all $1 \le i \le k$ and $1 \le l \le n$, and the interaction term f satisfies

$$f_i(v^1, \dots, \underbrace{u}_{i}, \dots, v^k) \le f_i(u^1, \dots, \underbrace{u}_{i}, \dots, u^k),$$

whenever $v^j \le u^j$, for $i \neq j$ and all $1 \le i, j \le k$.

Proof. Let u_0 and v_0 be given initial data satisfying the assumptions of Theorem 2.3 and assume $u_0 \succcurlyeq v_0$. We show that the order $\succcurlyeq$ is preserved by the corresponding solutions $u = u(\,\cdot\,, t; u_0)$ and $v = v(\,\cdot\,, t; v_0)$, that is $u \succcurlyeq v$ for $t > 0$, if and only if a, γ and f fulfill the stated conditions. Defining the function $w := u - v$ it satisfies

$$\begin{cases} \partial_t w = a(u) \cdot \Delta u - a(v) \cdot \Delta v & \\ \qquad\qquad -\gamma(u) \cdot Du + \gamma(v) \cdot Dv + f(u) - f(v) & \Omega \times (0, T] \\ w|_{t=0} = w_0 & \Omega \times \{0\} \\ w|_{\partial\Omega} = 0 & \partial\Omega \times [0, T]. \end{cases} \tag{4.7}$$

Necessity: We assume that the solutions satisfy $u \succcurlyeq v$, that is $w \in K^+$ for $t > 0$, and deduce the necessity of the stated conditions. Note that the initial data is non-negative $w_0 := u_0 - v_0 \in K^+$. Following the same arguments as in the first part of the proof of Theorem 2.3 leads to the inequality

$$\Big(a(u_0) \cdot \Delta u_0 - a(v_0) \cdot \Delta v_0 - \gamma(u_0) \cdot Du_0 + \gamma(v_0) \cdot Dv_0 + f(u_0) - f(v_0), \varphi \Big)_{L^2(\Omega; \mathbb{R}^k)} \ge 0$$

for an arbitrary $\varphi \in K^+$, which is orthogonal to w_0. The particular choice of the functions $u_0 = (\tilde{u}^1, \dots, \underbrace{\tilde{u}}_{i}, \dots, \tilde{u}^k)$ and $v_0 = (\tilde{v}^1, \dots, \underbrace{\tilde{u}}_{i}, \dots, \tilde{v}^k)$ with $\tilde{u}^j \ge \tilde{v}^j$

for $1 \le j \le k$, $j \neq i$, and an arbitrary $\varphi = (0, \ldots, \underbrace{\tilde{\varphi}}_{i}, \ldots, 0)$, with non-negative $\tilde{\varphi} \in L^2(\Omega)$ yields the pointwise estimate

$$\sum_{j=1, j\neq i}^{k} \left[a_{ij}(u_0) \cdot \Delta \tilde{u}^j - a_{ij}(v_0) \cdot \Delta \tilde{v}^j\right] + [a_{ii}(u_0) - a_{ii}(v_0)] \cdot \Delta \tilde{u}^i -$$
$$-\sum_{l=1}^{n} \sum_{j=1, j\neq i}^{k} \left[\gamma_{ij}^l(u_0) \cdot \partial_{x_l} \tilde{u}^j + \gamma_{ij}^l(v_0) \cdot \partial_{x_l} \tilde{v}^j\right] + \sum_{l=1}^{n} [\gamma_{ii}(u_0) - \gamma_{ii}(v_0)] \cdot \partial_{x_l} \tilde{u}^i +$$
$$+ f_i(u_0) - f_i(v_0) \ge 0 \qquad a.e. \text{ in } \Omega.$$

We deduce $a_{ij}(u^1, \ldots, u^k) = \gamma_{ij}(u^1, \ldots, u^k) = 0$, for all $1 \le i, j \le k$, $i \neq j$, and the diagonal coefficient functions necessarily satisfy

$$\begin{aligned} a_{ii}(u^1, \ldots, \underbrace{u}_{i}, \ldots, u^k) &= a_{ii}(v^1, \ldots, \underbrace{u}_{i}, \ldots, v^k) \\ \gamma_{ii}^l(u^1, \ldots, \underbrace{u}_{i}, \ldots, u^k) &= \gamma_{ii}^l(v^1, \ldots, \underbrace{u}_{i}, \ldots, v^k) \end{aligned}$$

for $1 \le l \le n$. Moreover, for the interaction term follow the monotonicity conditions

$$f_i(v^1, \ldots, \underbrace{u}_{i}, \ldots, v^k) \le f_i(u^1, \ldots, \underbrace{u}_{i}, \ldots, u^k),$$

whenever $v^j \le u^j$, $i \neq j$, for all $1 \le i, j \le k$.

Sufficiency: Under the stated assumptions on a, γ and f, system (4.7) takes the form

$$\begin{cases} \partial_t w^i = a_{ii}(u) \cdot \Delta u^i - a_{ii}(v) \cdot \Delta v^i - \sum_{l=1}^{n} \left[\gamma_{ii}^l(u) \cdot \partial_{x_l} u^i - \gamma_{ii}^l(v) \cdot \partial_{x_l} v^i\right] \\ \qquad\qquad + f_i(u) - f_i(v) \\ w|_{t=0} = w_0 \\ w|_{\partial\Omega} = 0 \end{cases}$$

for $1 \le i \le k$ with initial data $w_0 \in K^+$. Equivalently to the claim we prove for the corresponding solution $w = w(\,\cdot\,, t; w_0)$ that $(w_0^i)_- = 0$ implies $w_-^i = 0$ a.e. in Ω for $t > 0$ and all $1 \le i \le k$. Notice that the assumptions on the functions a_{ii} lead to the equality

$$\begin{aligned} a_{ii}(v) &= \int_0^1 \frac{d}{ds} a_{ii}(v^1, \ldots, sv^i + (1-s)u^i, \ldots, v^k) ds + a_{ii}(v^1, \ldots, u^i, \ldots, v^k) = \\ &= (v^i - u^i) \int_0^1 \partial_i a_{ii}(v^1, \ldots, sv^i + (1-s)u^i, \ldots, v^k) ds + \\ &\quad + a_{ii}(u^1, \ldots, u^i, \ldots, u^k), \end{aligned}$$

for all $1 \leq i \leq k$. Hence, we obtain

$$\begin{aligned}
& a_{ii}(u) \cdot \Delta u^i - a_{ii}(v) \cdot \Delta v^i = \\
= \ & a_{ii}(u) \cdot \Delta w^i + w^i \cdot \int_0^1 \partial_i a_{ii}(v^1, \ldots, sv^i + (1-s)u^i, \ldots, v^k) ds \cdot \Delta v^i = \\
= \ & a_{ii}(u) \cdot \Delta w^i + w^i \cdot \tilde{a}_{ii}(v^1, \ldots, sv^i + (1-s)u^i, \ldots, v^k) \cdot \Delta v^i
\end{aligned}$$

and, analogously, for the functions γ_{ii}^l follows

$$\begin{aligned}
& \gamma_{ii}^l(u) \cdot \partial_{x_l} u^i - \gamma_{ii}^l(v^1, \ldots, v^k) \cdot \partial_{x_l} v^i = \\
= \ & \gamma_{ii}^l(u) \cdot \partial_{x_l} w^i + w^i \cdot \int_0^1 \partial_i \gamma_{ii}^l(v^1, \ldots, sv^i + (1-s)u^i, \ldots, v^k) ds \cdot \partial_{x_l} v^i = \\
= \ & \gamma_{ii}^l(u) \cdot \partial_{x_l} w^i + w^i \cdot \tilde{\gamma}_{ii}^l(v^1, \ldots, sv^i + (1-s)u^i, \ldots, v^k) \cdot \partial_{x_l} v^i,
\end{aligned}$$

for all $1 \leq l \leq n$ and $1 \leq i \leq k$. Multiplying the i-th equation by w_-^i and integrating over Ω yields

$$\begin{aligned}
-\partial_t \|w_-^i\|_{L^2(\Omega)}^2 \ = \ & - \int_\Omega a_{ii}(u) \cdot \Delta w_-^i \cdot w_-^i dx - \\
& - \int_\Omega (w_-^i)^2 \cdot \tilde{a}_{ii}(v^1, \ldots, sv^i + (1-s)u^i, \ldots, v^k) \cdot \Delta v^i dx + \\
& + \sum_{l=1}^n \int_\Omega \Big[\gamma_{ii}^l(u) \cdot \partial_{x_l} w_-^i \cdot w_-^i \\
& + (w_-^i)^2 \cdot \tilde{\gamma}_{ii}^l(v^1, \ldots, sv^i + (1-s)u^i, \ldots, v^k) \cdot \partial_{x_l} v^i \Big] dx + \\
& + \int_\Omega (f_i(u) - f_i(v)) \cdot w_-^i dx.
\end{aligned}$$

Taking into account the homogeneous Dirichlet boundary conditions we derive for the first term of the right-hand side

$$\begin{aligned}
- \ & \int_\Omega a_{ii}(u) \cdot \Delta w_-^i \cdot w_-^i dx = \int_\Omega \nabla\big((a_{ii}(u) \cdot w_-^i\big) \cdot \nabla w_-^i dx = \\
= \ & \int_\Omega a_{ii}(u) \cdot |\nabla w_-^i|^2 dx + \sum_{j=1}^k \int_\Omega \partial_j a_{ii}(u) \cdot \nabla u^j \cdot w_-^i \cdot \nabla w_-^i dx.
\end{aligned}$$

Moreover, the second integral can be estimated by

$$\sum_{j=1}^k \int_\Omega |\partial_j a_{ii}(u) \cdot \nabla u^j \cdot w_-^i \cdot \nabla w_-^i| dx \leq c_1 \int_\Omega \sum_{l=1}^n |\partial_{x_l} w_-^i| \cdot |w_-^i| dx,$$

for some constant $c_1 \geq 0$. Here, we used the assumption $a_{ii} \in C^1(\mathbb{R}^k)$ and the fact that u^j and $|\nabla u^j|$ are in $L^\infty(\Omega)$. For the second term we obtain

$$\begin{aligned}
& -\int_\Omega (w^i_-)^2 \cdot \tilde{a}_{ii}(v^1, \ldots, sv^i + (1-s)u^i, \ldots, v^k) \cdot \Delta v^i dx = \\
= \quad & -2\int_\Omega \nabla w^i_- \cdot w^i_- \cdot \tilde{a}_{ii}(v^1, \ldots, sv^i + (1-s)u^i, \ldots, v^k) \cdot \nabla v^i dx - \\
& -\int_\Omega (w^i_-)^2 \cdot \sum_{j=1, j\neq i}^k \int_0^1 \partial_j \partial_i a_{ii}(v^1, \ldots, sv^i + (1-s)u^i, \ldots, v^k) ds \cdot \nabla v^j \cdot \nabla v^i dx - \\
& -\int_\Omega (w^i_-)^2 \cdot \sum_{j=1}^k \int_0^1 \partial_i^2 a_{ii}(v^1, \ldots, sv^i + (1-s)u^i, \ldots, v^k) \cdot s ds \nabla (v^i - u^i) \cdot \nabla v^i dx \leq \\
\leq \quad & c_2 \int_\Omega \sum_{l=1}^n |\partial_{x_l} w^i_-| \cdot |w^i_-| dx + c_3 \|w^i_-\|^2_{L^2(\Omega)},
\end{aligned}$$

for some constants $c_2, c_3 \geq 0$. Again, we used that u, v, $|\nabla u|$ and $|\nabla v|$ belong to the space $L^\infty(\Omega)$ and the assumption on the partial derivatives of the functions a_{ii}. Similarly, we estimate the convection terms

$$\begin{aligned}
& \left| -\sum_{l=1}^n \int_\Omega \gamma^l_{ii}(u) \cdot \partial_{x_l} w^i_- \cdot w^i_- + (w^i_-)^2 \cdot \tilde{\gamma}^l_{ii}(v^1, \ldots, sv^i + (1-s)u^i, \ldots, v^k) \cdot \partial_{x_l} v^i dx \right| \\
& \qquad \leq c_4 \cdot \int_\Omega \sum_{l=1}^n |\partial_{x_l} w^i_-| \cdot |w^i_-| dx + c_5 \|w^i_-\|^2_{L^2(\Omega)},
\end{aligned}$$

for some constants $c_4, c_5 \geq 0$, where we used that $\gamma^l_{ii} \in C^1(\mathbb{R}^k)$ for $1 \leq l \leq n$ as well as the boundedness of u, v and $|\nabla v|$ on Ω. Finally, we observe

$$\begin{aligned}
f_i(u) - f_i(v) \quad = \quad & f_i(u) - f_i(v^1, \ldots, u^i, \ldots, v^k) + \\
& + w^i \cdot \int_0^1 \partial_i f_i(v^1, \ldots, sv^i + (1-s)u^i, \ldots, v^k) ds.
\end{aligned}$$

Hence, for the remaining integral follows

$$\begin{aligned}
& - \int_\Omega (f_i(u) - f_i(v)) \cdot w^i_- dx = \\
&= -\int_\Omega \big(f_i(u^1,\dots,u^k) - f_i(v^1,\dots,u^i,\dots,v^k)\big) \cdot w^i_- dx - \\
& \quad - \int_\Omega \int_0^1 \partial_i f_i(v^1,\dots, sv^i + (1-s)u^i, \dots, v^k) ds \cdot (w^i_-)^2 dx \le \\
&\le \Big| - \int_\Omega \int_0^1 \partial_i f_i(v^1,\dots, sv^i + (1-s)u^i, \dots, v^k) ds \cdot (w^i_-)^2 dx \Big| \le \\
&\le c_6 \|w^i_-\|^2_{L^2(\Omega)},
\end{aligned}$$

for some constant $c_6 \ge 0$. We used that $f_i \in C^1(\mathbb{R}^k)$ and the assumption

$$f_i(v^1,\dots,\underbrace{u}_{i},\dots,v^k) \le f_i(u^1,\dots,\underbrace{u}_{i},\dots,u^k).$$

This is a priori only true for $v^j \le u^j$ for $i \ne j$, $1 \le j \le k$. However, similar arguments as at the end of the proof of Theorem 3.2 justify this inequality. Summing up and estimating all integrals of the form $\int_\Omega \sum_{l=1}^n |\partial_{x_l} w^i_-| \cdot |w^i_-| dx$ by Young's inequality we deduce

$$\partial_t \|w^i_-\|^2_{L^2(\Omega)} \le C \cdot \|w^i_-\|^2_{L^2(\Omega)},$$

for some constant $C \ge 0$. Applying Gronwall's lemma and using the assumption $(w^i_0)_- = 0$ we conclude $w^i_- = 0$ a.e. in Ω. This proves that the order is preserved by the solutions u and v for $t > 0$. $\square$

A direct consequence of this result is the generalization for an arbitrary order relation. We formulate necessary and sufficient conditions for the system of quasi-linear equations to be order preserving with respect to the order $\precsim$.

Theorem 4.6. *In addition to the hypothesis of* Theorem 2.3 *we assume that the second partial derivatives of the diagonal coefficient functions exist and belong to the space* $L^\infty_{loc}(\mathbb{R}^k)$. *Then, system* (2.1) *is order preserving with respect to* $\precsim$ *if and only if the matrices* a *and* γ^l *are diagonal, for the diagonal coefficient functions*

$$\begin{aligned}
a_{ii}(u^1,\dots,\underbrace{u}_{i},\dots,u^k) &= a_{ii}(v^1,\dots,\underbrace{u}_{i},\dots,v^k), \\
\gamma^l_{ii}(u^1,\dots,\underbrace{u}_{i},\dots,u^k) &= \gamma^l_{ii}(v^1,\dots,\underbrace{u}_{i},\dots,v^k)
\end{aligned}$$

holds for all $1 \le l \le n$ *and* $1 \le i \le k$, *and the interaction term satisfies*

$$\begin{aligned}
f_i(v^1,\dots,\underbrace{u^i}_{i},\dots,v^k) &\le f_i(u^1,\dots,\underbrace{u^i}_{i},\dots,u^k) \quad \text{if } i \in \sigma_1, \\
f_i(v^1,\dots,\underbrace{u^i}_{i},\dots,v^k) &\ge f_i(u^1,\dots,\underbrace{u^i}_{i},\dots,u^k) \quad \text{if } i \in \sigma_2,
\end{aligned}$$

whenever $v^j \le u^j$, for $j \in \sigma_1$ and $u^j \le v^j$ for $j \in \sigma_2$, $j \neq i$.

Proof. Let u_0 and v_0 be given initial data satisfying the assumptions of Theorem 2.3 and assume $u_0 \succsim v_0$. We show that by the corresponding solutions $u = u(\,\cdot\,, t; u_0)$ and $v = v(\,\cdot\,, t; v_0)$ the order is preserved, that is $u \succsim v$ for $t > 0$, if and only if a, γ and f fulfill the stated conditions. Defining w by

$$w^i := \begin{cases} u^i - v^i & \text{if } i \in \sigma_1 \\ -(u^i - v^i) & \text{if } i \in \sigma_2 \end{cases}$$

it satisfies the system

$$\begin{cases} \partial_t w = \tilde{a}(u) \cdot \Delta u - \tilde{a}(v) \cdot \Delta v - \tilde{\gamma}(u) \cdot Du + \tilde{\gamma}(v) \cdot Dv + F(w) \\ w|_{t=0} = w_0 \\ w|_{\partial\Omega} = 0, \end{cases}$$

where $w_0 \in K^+$ and the function F is defined by

$$F_i(w) := \begin{cases} f_i(u) - f_i(v) & i \in \sigma_1 \\ -(f_i(u) - f_i(v)) & i \in \sigma_2 \end{cases}$$

for $1 \le i \le k$. Moreover, the coefficient functions of the diffusion matrix $\tilde{a}$ are given by

$$\tilde{a}_{ij}(u) := \begin{cases} a_{ij}(u) & \text{if } j \in \sigma_1 \\ -a_{ij}(u) & \text{if } j \in \sigma_2 \end{cases}$$

and of the convection matrix by

$$\tilde{\gamma}^l_{ij}(u) := \begin{cases} \gamma^l_{ij}(u) & \text{if } j \in \sigma_1 \\ -\gamma^l_{ij}(u) & \text{if } j \in \sigma_2 \end{cases}$$

for all $1 \le i, j \le k$ and $1 \le l \le n$. By Theorem 4.5 we conclude for the corresponding solution $w = w(\,\cdot\,, t; w_0)$ that $w_0 \succcurlyeq 0$ implies $w \succcurlyeq 0$ for $t > 0$ if and only if the matrices a and γ^l satisfy $a_{ij}(u) = \gamma_{ij}(u) = 0$, for $i \neq j$, $1 \le i, j \le k$, moreover

$$\begin{aligned} a_{ii}(u^1, \ldots, \underbrace{u}_{i}, \ldots, u^k) &= a_{ii}(v^1, \ldots, \underbrace{u}_{i}, \ldots, v^k), \\ \gamma^l_{ii}(u^1, \ldots, \underbrace{u}_{i}, \ldots, u^k) &= \gamma^l_{ii}(v^1, \ldots, \underbrace{u}_{i}, \ldots, v^k), \end{aligned}$$

for $1 \leq l \leq n$, and the interaction term f satisfies

$$F_i(v^1, \dots, \underbrace{u}_{i}, \dots, v^k) \leq F_i(u^1, \dots, \underbrace{u}_{i}, \dots, u^k),$$

whenever $w^j \geq 0$ for $1 \leq i, j \leq k$, $i \neq j$. By the definition of w, the claim that $w_0 \succcurlyeq 0$ implies $w \succcurlyeq 0$ for $t > 0$ is equivalent to the claim that system (2.1) is order preserving with respect to $\precsim$. □

5 Generalized Boundary Conditions

In this section we discuss whether Theorems 2.3 and 3.2 remain valid under more general boundary conditions. Recall that we always assumed homogeneous Dirichlet conditions. It turns out that for semi-linear systems the positivity criterion remains identical for all relevant boundary values of the solution.

5.1 Semi-Linear Case

5.1.1 Inhomogeneous Dirichlet Boundary Conditions

Instead of homogeneous Dirichlet boundary conditions we assume the solution u of the semi-linear system (3.4) satisfies

$$u \mid_{\partial\Omega} = g,$$

where $g : \partial\Omega \to \mathbb{R}^d$ is a given continuous non-negative function. Notice that the boundary conditions were not used in the first part of the proof of Theorem 3.2. Hence, the necessity of the stated conditions remains unchanged. Moreover, it is certainly natural to require non-negative boundary data. In order to prove the sufficiency it is enough to estimate the boundary integrals we obtain through integration by parts. Estimating the diffusion term we now get

$$\begin{aligned}
&(a_i \cdot \Delta u^i_-, u^i_-)_{L^2(\Omega)} \\
&\quad = \int_{\partial\Omega} \frac{\partial}{\partial\nu}(a_i \cdot u^i_-) \cdot u^i_- dx - \int_{\Omega} a_i \cdot |\nabla u^i_-|^2 dx = \\
&\quad = \int_{\partial\Omega} a_i \cdot (\frac{\partial}{\partial\nu} u^i_-) \cdot g^i_- dx - \int_{\Omega} a_i \cdot |\nabla u^i_-|^2 dx = -a_i \int_{\Omega} |\nabla u^i_-|^2 dx,
\end{aligned}$$

where we used that g is non-negative, which implies $u^i_- \mid_{\partial\Omega} = g^i_- = 0$. Consequently, the proof remains the same as in the case of homogeneous Dirichlet boundary conditions.

5.1.2 Neumann Boundary Conditions

Next, we assume u satisfies homogeneous Neumann boundary conditions

$$\frac{\partial}{\partial \nu} u \mid_{\partial \Omega} = 0.$$

Representing $u = u_+ - u_-$, that is $\frac{\partial}{\partial \nu} u^i = \frac{\partial}{\partial \nu} u^i_+ - \frac{\partial}{\partial \nu} u^i_- = 0$, follows $\frac{\partial}{\partial \nu} u^i_+ = -\frac{\partial}{\partial \nu} u^i_-$. This implies

$$\begin{aligned}
&(\not{a}_i \cdot \Delta u^i_-, u^i_-)_{L^2(\Omega)} \\
&\quad = \int_{\partial \Omega} a_i \cdot (\frac{\partial}{\partial \nu} u^i_-) \cdot u^i_- dx - \int_\Omega a_i \cdot |\nabla u^i_-|^2 dx = \\
&\quad = \int_{\partial \Omega} a_i \cdot (-\frac{\partial}{\partial \nu} u^i_+) \cdot u^i_- dx - \int_\Omega a_i \cdot |\nabla u^i_-|^2 dx = -a_i \int_\Omega |\nabla u^i_-|^2 dx
\end{aligned}$$

and the proof remains identical. In case of inhomogeneous Neumann conditions

$$\frac{\partial}{\partial \nu} u \mid_{\partial \Omega} = g,$$

we assume the function g is continuous and non-negative, that is $g_i \geq 0$ for all $1 \leq i \leq k$. This implies $\frac{\partial}{\partial \nu} u^i = \frac{\partial}{\partial \nu} u^i_+ - \frac{\partial}{\partial \nu} u^i_- = (g_i)_+ - (g_i)_- = (g_i)_+$ and therefore, $\frac{\partial}{\partial \nu} u^i_- = \frac{\partial}{\partial \nu} u^i_+ - (g_i)_+$ holds on the boundary $\partial \Omega$. Estimating the diffusion term we obtain

$$\begin{aligned}
&(\not{a}_i \cdot \Delta u^i_-, u^i_-)_{L^2(\Omega)} \\
&\quad = \int_{\partial \Omega} a_i \cdot (\frac{\partial}{\partial \nu} u^i_-) \cdot u^i_- dx - \int_\Omega a_i \cdot |\nabla u^i_-|^2 dx = \\
&\quad = \int_{\partial \Omega} a_i \cdot (\frac{\partial}{\partial \nu} u^i_+ - (g_i)_+) \cdot u^i_- dx - \int_\Omega a_i \cdot |\nabla u^i_-|^2 dx = \\
&\quad = -a_i \int_{\partial \Omega} (g_i)_+ \cdot u^i_- dx - \int_\Omega a_i \cdot |\nabla u^i_-|^2 dx \leq -\int_\Omega a_i \cdot |\nabla u^i_-|^2 dx.
\end{aligned}$$

Hence, Theorem 3.2 remains valid for solutions satisfying Neumann boundary conditions with non-negative function g.

5.1.3 Robin Boundary Conditions

Finally, we assume u satisfies mixed boundary conditions

$$\alpha u + \beta \frac{\partial}{\partial \nu} u \Big|_{\partial \Omega} = 0 \quad \text{with constants } \alpha, \beta > 0.$$

It follows $-\frac{\alpha}{\beta}(u^i_+ - u^i_-) = \frac{\partial}{\partial\nu}u^i_+ - \frac{\partial}{\partial\nu}u^i_-$. Therefore, computing the boundary integral we obtain

$$\begin{aligned}\int_{\partial\Omega}\frac{\partial}{\partial\nu}(a_i\cdot u^i_-)\cdot u^i_- dx &= \int_{\partial\Omega} a_i\cdot(\frac{\partial}{\partial\nu}u^i_-)\cdot u^i_- dx = \\ &= \int_{\partial\Omega} a_i\cdot(\frac{\alpha}{\beta}(u^i_+ - u^i_-) + \frac{\partial}{\partial\nu}u^i_+)\cdot u^i_- dx = \\ &= -a_i\int_{\partial\Omega}\frac{\alpha}{\beta}\cdot|u^i_-|^2 dx \le 0.\end{aligned}$$

This term can be omitted in the estimate of the diffusion term and the proof remains valid. For inhomogeneous Robin conditions

$$\alpha u + \beta\frac{\partial}{\partial\nu}u\big|_{\partial\Omega} = g, \quad \text{with } \alpha,\beta > 0,$$

we assume the function g is continuous and non-negative. It follows $\alpha(u^i_+ - u^i_-) + \beta(\frac{\partial}{\partial\nu}u^i_+ - \frac{\partial}{\partial\nu}u^i_-) = (g_i)_+ - (g_i)_- = (g_i)_+$ and consequently,

$$\frac{\partial}{\partial\nu}u^i_- = \frac{\alpha}{\beta}(u^i_+ - u^i_-) + \frac{\partial}{\partial\nu}u^i_+ - \frac{1}{\beta}(g_i)_+$$

holds on the boundary. This implies

$$\begin{aligned}\int_{\partial\Omega}\frac{\partial}{\partial\nu}(a_i\cdot u^i_-)\cdot u^i_- dx &= \int_{\partial\Omega} a_i\cdot\big(\frac{\alpha}{\beta}(u^i_+ - u^i_-) + \frac{\partial}{\partial\nu}u^i_+ - \frac{1}{\beta}(g_i)_+\big)\cdot u^i_- dx = \\ &= \int_{\partial\Omega} a_i\cdot(\frac{\alpha}{\beta}(-u^i_-) - \frac{1}{\beta}(g_i)_+)\cdot u^i_- dx = \\ &= -a_i\cdot\frac{\alpha}{\beta}\int_{\partial\Omega}|u^i_-|^2 dx - a_i\cdot\frac{1}{\beta}\int_{\partial\Omega}(g_i)_+\cdot u^i_- dx \le 0,\end{aligned}$$

and the proof continues as in the case of homogeneous Dirichlet conditions.

5.2 Quasi-linear Systems

Finally, we discuss inhomogeneous boundary data for the solution of the quasi-linear system (2.1). The boundary integral in the proof of Theorem 2.3 yields

$$\begin{aligned}\int_{\partial\Omega}\frac{\partial}{\partial\nu}(a_{ii}(u)\cdot u^i_-)\cdot u^i_- dx &= \sum_{j=1}^{k}\int_{\partial\Omega}|u^i_-|^2\partial_j a_{ii}(u)\nabla u^j\cdot\nu dx + \\ &\quad + \int_{\partial\Omega}u^i_- a_{ii}(u)\nabla u^i_-\cdot\nu dx. \qquad (5.8)\end{aligned}$$

If we do not impose further restrictions on the diagonal diffusion coefficient functions a_{ii}, the proof remains valid for **inhomogeneous Dirichlet boundary conditions**

$$u|_{\partial\Omega} = g,$$

with non-negative g. Indeed, if the boundary data satisfies $g^i \geq 0$ for all $1 \leq i \leq k$, it follows $u^i_- = 0$ on the boundary and both integrals in (5.8) vanish. The same applies in case of **homogeneous Neumann conditions**

$$\frac{\partial u^i}{\partial \nu}|_{\partial\Omega} = 0.$$

Here we deduce $\frac{\partial}{\partial\nu} u^i_+ = -\frac{\partial}{\partial\nu} u^i_-$ and the integrals in (5.8) are zero as the supports of u_+ and u_- are disjoint.

6 Further Generalizations

The method used to prove the sufficiency of the stated conditions in Theorem 2.3 is applicable for various, more general parabolic systems. In order to illustrate this, we consider the degenerate parabolic system

$$\begin{cases} \partial_t u = a \cdot \Delta\Phi(u) + F(u) & \Omega \times (0,T] \\ u|_{t=0} = u_0 & \Omega \times \{0\} \\ u|_{\partial\Omega} = 0 & \partial\Omega \times [0,T], \end{cases}$$

where the density dependent diffusion term is given by

$$\Delta\Phi(u) = \big(\Delta(u^1)^m, \ldots, \Delta(u^k)^m\big)$$

with $m \in \mathbb{N}$, $m > 2$. As before, u^i, $1 \leq i \leq k$, denotes the i-th component of the vector-valued function $u = (u^1, \ldots, u^k)$. We claim that the solution u corresponding to non-negative initial data u_0 remains non-negative, if and only if the matrix a is diagonal and F satisfies

$$F_i(u^1, \ldots, \underbrace{0}_{i}, \ldots, u^k) \geq 0 \quad \text{for } u^1 \geq 0, \ldots, u^k \geq 0.$$

The necessity of these conditions can be obtained exactly as in the proof of Theorem 2.3. In order to show that these assumptions are sufficient we assume a is diagonal, $a = diag(a_1, \ldots, a_k)$. The system then takes the form

$$\partial_t u^i = a_i \cdot \Delta((u^i)^m) + F_i(u^1, \ldots, u^k).$$

Multiplying this equation by u^i_- and integrating over Ω leads to

$$\begin{aligned}-\frac{1}{2}\partial_t\|u^i\|^2 &= -a_i\int_\Omega \nabla((u^i)^m)\cdot\nabla u^i_- dx + \int_\Omega F_i(u^1,\dots,u^k)\cdot u^i_- dx = \\ &= -a_i\cdot m\cdot(-1)^{m-1}\int_\Omega (u^i_-)^{m-1}\cdot|\nabla u^i_-|^2 dx + \int_\Omega F_i(u^1,\dots,u^k)\cdot u^i_- dx,\end{aligned}$$

where we took into account the homogeneous Dirichlet boundary conditions. Therefore, it follows

$$\begin{aligned}&\frac{1}{2}\partial_t\|u^i\|^2 = \\ &= a_i\cdot m\cdot(-1)^{m-1}\int_\Omega (u^i_-)^{m-1}\cdot|\nabla u^i_-|^2 dx - \int_\Omega F_i(u^1,\dots,u^k)\cdot u^i_- dx. \quad (6.9)\end{aligned}$$

The last integral can be estimated analogously as in the proof of Theorem 2.3,

$$-\int_\Omega F_i(u^1,\dots,u^k)\cdot u^i_- dx \le C\cdot\|u^i_-\|^2.$$

If m is odd, the first term in (6.9) can be omitted and the claim follows immediately by Gronwall's Lemma and the assumption $(u^i_0)_- = 0$. Otherwise, for even $m > 2$, we may assume that u^i_- and ∇u^i_- are in $L^\infty(\Omega)$ to obtain an estimate of the form

$$\partial_t\|u^i\|^2 \le c\cdot\|u^i\|^2,$$

for some constant $c \ge 0$. As before, we conclude $u^i_- = 0$ a.e. in Ω.

7 Applications

7.1 Chemotaxis

The Keller-Segel model describes the dynamics of a population in a domain Ω following the gradient of a chemotactic agent, which is produced by the population itself. Based on this model, the following parabolic system was analyzed by Jäger and Luckhaus in [9]. The population density u and concentration of the chemotactic agent v satisfy

$$\begin{cases} u_t = \Delta u - \chi\cdot\nabla(u\nabla v) & \text{in } \Omega\times(0,\infty) \\ v_t = \Delta v - (u-1) & \text{in } \Omega\times(0,\infty) \\ \frac{\partial u}{\partial\nu} = \frac{\partial v}{\partial\nu} = 0 & \text{on } \partial\Omega\times[0,\infty) \\ u(\cdot,0) = u_0,\quad v(\cdot,0) = v_0 & \text{in } \Omega\times\{0\}, \end{cases} \quad (7.10)$$

where χ is a positive constant. Moreover, $\Omega \subset \mathbb{R}^2$ is assumed to be open and bounded with C^1-boundary. The initial data $u_0, v_0 \in C^1(\overline{\Omega})$ satisfy the boundary conditions and $u_0, v_0 \geq 0$. Note that the first equation of (7.10) can be written as

$$u_t = \Delta u + \chi \cdot \nabla u \cdot \nabla v + \chi \cdot u \Delta v. \tag{7.11}$$

The cross-diffusion term is of the form required for the positivity of the solution u in Theorem 2.3. Moreover, we see that the proof of Theorem 2.3 can be generalized to system (7.10). Indeed, if we multiply the second term in (7.11) by u_-, integrate over Ω and use Young's inequality, we obtain the estimate

$$|\chi \int_\Omega \nabla u \cdot \nabla v \cdot u_- dx| \leq \epsilon \int_\Omega |\nabla u_-|^2 dx + C \int_\Omega |u_-|^2 dx$$

for some constant $C \geq 0$. Under the assumption that u, v and their gradients satisfy L^∞-estimates Theorem 2.3 implies that the density u is non-negative. Furthermore, if the density u is bounded by 1, the positivity of the concentration v follows as the interaction term in the second equation then satisfies $-(u-1) \geq 0$.

Remark 7.1. *Our criterion applies in the same way to so-called chemotaxis growth models. See [12] for instance, where a model to study aggregating patterns of bacteria due to chemotaxis and growth was presented.*

7.2 A System of PDEs Including the Effect of Porous Medium and Chemotaxis

Let $\Omega \subset \mathbb{R}^n$ be open and bounded with C^1-boundary $\partial\Omega$. The following type of equations arises in the modelling of biomass spreading mechanisms via chemotaxis

$$\begin{cases} M_t = \nabla(M^\alpha \cdot \nabla M) - \nabla(M^\gamma \cdot \nabla \rho) + f(M, \rho) & \text{in } \Omega \times (0, \infty) \\ \rho_t = \Delta \rho - g(M, \rho) & \text{in } \Omega \times (0, \infty) \\ M = 0, \quad \rho = 1 & \text{on } \partial\Omega \times [0, \infty) \\ M(\cdot, 0) = M_0, \quad \rho(\cdot, 0) = \rho_0 & \text{in } \Omega \times \{0\}, \end{cases} \tag{7.12}$$

with smooth initial data satisfying the boundary conditions and $M_0, \rho_0 \geq 0$. The constants α and γ are assumed to fulfill $\gamma + 1 \leq \alpha \leq 2\gamma + 2$ and $\gamma + 1 \leq \alpha$ (which implies $\alpha \geq 4, \gamma \geq 3$). This system was analyzed in [6], where the well-posedness is proved for the dimensions $n = 1, 2, 3$.

First, we apply Theorem 2.3 to derive conditions on the function g that yield the positivity of the density ρ. According to our criterion and the results of Section 5 the solution $\rho(\cdot, \cdot; \rho_0)$ corresponding to initial data $\rho_0 \geq 0$ remains non-negative if and only if the interaction term satisfies

$$g(M, 0) \leq 0 \quad \text{whenever } M \geq 0. \tag{7.13}$$

Note that the L^∞-property of the solutions was essential for the proof of Theorem 2.3. However, we may apply Theorem 2.3 to additionally deduce a necessary condition on the interaction term g for the boundedness of ρ. Defining the function $\tilde{\rho} := 1 - \rho$, the density ρ is bounded by 1 if and only if $\tilde{\rho}$ is non-negative. By Theorem 2.3, this is the case if and only if

$$g(M, 1) \geq 0 \quad \text{whenever } M \geq 0. \tag{7.14}$$

Indeed, the function $\tilde{\rho}$ satisfies

$$\tilde{\rho}_t = \Delta\tilde{\rho} + g(M, \rho) \quad \text{in } \Omega \times (0, \infty)$$

with homogeneous boundary conditions $\tilde{\rho}|_{\partial\Omega} = 0$. The interaction term g in [6] is of the form

$$g(M, \rho) = g_0(\rho)M + c_1\rho,$$

where the constant $c_1 > 0$, the function g_0 satisfies $g_0(0) = 0$ and

$$0 \leq g_0(\rho) \leq c_2 \quad \text{for some constant } c_2 > 0 \text{ and } \rho \geq 0.$$

Hence, conditions (7.13) and (7.14) are clearly satisfied. Furthermore, by Theorem 2.3, the density M remains non-negative if the interaction term satisfies $f(0, \rho) \geq 0$ for all $\rho \geq 0$. This is certainly true due to the assumption

$$-f_1 M^2 \leq f(M, \rho) \leq f_2 M - f_3 M^2,$$

which was made in [6].

7.3 Biofilm Model

We assume $\Omega \subset \mathbb{R}^n$ is an open and bounded subset with C^1-boundary $\partial\Omega$. The following non-linear density-dependent system of reaction-diffusion equations describes the spacial spreading of biomass during the development of microbial films. The function C represents the substrate concentration and M the biomass density, they satisfy

$$\begin{cases} M_t = d_1 \nabla\big(D_M(M)\nabla M\big) + k_3 \frac{MC}{k_2 + |C|} - k_4 M & \text{in } \Omega \times (0, T) \\ C_t = d_2 \Delta C - k_1 \frac{CM}{k_2 + |C|} & \text{in } \Omega \times (0, T) \\ C = 1, \quad M = 0 & \text{on } \partial\Omega \times [0, T] \\ C(\cdot, 0) = C_0, \quad M(\cdot, 0) = M_0 & \text{in } \Omega \times \{0\}, \end{cases} \tag{7.15}$$

where the constants $d_1, d_2 > 0$ and $k_1, k_2, k_3, k_4 \geq 0$. The density-dependent diffusion coefficient for the biomass M is given by

$$D_M(M) := \frac{|M|^a}{(1 - M)^b} \nabla M$$

with exponents $a, b \geq 1$. Moreover, the initial data is assumed to be smooth, satisfies the boundary conditions and $u_0, v_0 \geq 0$. Theorem 2.3 yields the positivity of the solutions.

Indeed, the existence of solutions of system (7.15) was proved in [1] considering the non-degenerate auxiliary system

$$\begin{cases} M_t = d_1 \nabla(F_\epsilon(M)\nabla M) + k_3 \frac{MC}{k_2+|C|} - k_4 M & \text{in } \Omega \times (0,T) \\ C_t = d_2 \Delta C - k_1 \frac{CM}{k_2+|C|} & \text{in } \Omega \times (0,T) \\ C = 1, \quad M = 0 & \text{on } \partial\Omega \times [0,T] \\ C(\cdot,0) = C_0, \quad M(\cdot,0) = M_0 & \text{in } \Omega \times \{0\}, \end{cases} \tag{7.16}$$

where the function F_ϵ is given by

$$F_\epsilon(z) := \begin{cases} \frac{|z+\epsilon|^b}{(1-z)^a} & z \leq 1-\epsilon \\ z^a & z > 1-\epsilon, \end{cases}$$

for $0 < \epsilon < 1$. This system is regular parabolic and quasi-linear without cross-diffusion terms. Using a comparison theorem for parabolic systems it was proofed in [1] that the solutions M^ϵ and C^ϵ of the auxiliary system satisfy $C^\epsilon, M^\epsilon \leq 1$ and consequently $0 \leq F_\epsilon(M) \leq 1$ holds. The function C^ϵ satisfies Dirichlet boundary conditions $C^\epsilon|_{\partial\Omega} = 1$. Hence, by Theorem 2.3 and the results of Section 5 the solution $C^\epsilon(\cdot, t; C_0)$ remains non-negative for $t > 0$ as the interaction term

$$g(M,C) := -k_1 \frac{CM}{k_2 + |C|}$$

satisfies $g(M, 0) \geq 0$. The positivity of M^ϵ follows as well from Theorem 2.3 as

$$f(M,C) := k_3 \frac{MC}{k_2 + |C|} - k_4 M$$

satisfies $f(0, C) \geq 0$. However, recalling the proof of Theorem 2.3 we see that the positivity of M^ϵ and C^ϵ can be deduced without the L^∞-assumption on the solutions. Indeed, we multiply the first equation of (7.16) by M^ϵ_- and integrate over Ω, which leads to the following estimate for the interaction term

$$|\int_\Omega f(C^\epsilon, M^\epsilon) M^\epsilon_- dx| \leq k_3 \int_\Omega \left| \frac{C^\epsilon}{k_2 + |C^\epsilon|} \right| \cdot |M^\epsilon_-|^2 dx + k_4 \int_\Omega |M^\epsilon_-|^2 dx \leq c \|M^\epsilon_-\|_{L^2(\Omega)},$$

for some constant $c \geq 0$. This implies the positivity of the biomass density M^ϵ. Next, multiplying the second equation of (7.16) by C^ϵ_- and integrating over Ω yields

$$\frac{1}{2} \frac{\partial}{\partial_t} \int_\Omega |C^\epsilon_-|^2 dx \leq -k_1 \int_\Omega \frac{M^\epsilon}{k_2 + |C^\epsilon|} \cdot |C^\epsilon_-|^2 dx \leq 0,$$

where we used the positivity of M^ϵ. This certainly implies that $C^\epsilon_- = 0$ a.e. in Ω. It was shown that the solutions M^ϵ and C^ϵ converge to the solutions M and C of the original system in $C_{loc}(\mathbb{R}^+; L^2(\Omega))$. This proves the positivity of the solutions M and C of the original system (7.15) (cf. [1]).

7.4 Quorum Sensing

Finally, we show the positivity of solutions of a mathematical model describing quorum sensing in biofilm communities. Quorum sensing is a mechanism of cell communication to coordinate behavior in groups and gene production by the production of extracellular signalling molecules. The dependent variables of the model are the density of the signaling molecule A, the concentration of the growth limiting nutrient C, the down-regulated biomass density M^0 and the up-regulated biomass fraction M^1. Cells constantly produce and release signalling molecules, when a critical concentration of the signalling molecule is reached, the cells undergo changes. These cells are then called up-regulated and produce the signalling molecule at an increased rate. The following model was analyzed in [7]

$$\begin{cases} M^0_t = \nabla(D_M(M)\nabla M^0) & \\ \qquad\qquad +k_3\frac{M^0C}{k_2+C} - k_4M^0 - k_5A^nM^0 + k_5M^1 & \text{in } \Omega\times(0,T) \\ M^1_t = \nabla(D_M(M)\nabla M^1) & \\ \qquad\qquad +k_3\frac{M^1C}{k_2+C} - k_4M^1 + k_5A^nM^0 - k_5M^1 & \text{in } \Omega\times(0,T) \\ C_t = d_1\Delta C - k_1\frac{CM}{k_2+C} & \text{in } \Omega\times(0,T) \\ A_t = d_2\Delta A - \gamma A + \alpha M^0 + (\alpha+\beta)M^1 & \text{in } \Omega\times(0,T) \\ M^0 = 0, \quad M^1 = 0, \quad C = 1, \quad A = 1, & \text{on } \partial\Omega\times[0,T] \\ M^0 = M^0_0, \quad M^1 = M^1_0, \quad C = C_0, \quad A = A_0, & \text{in } \Omega\times\{0\}, \end{cases} \tag{7.17}$$

where $M := M^0+M^1$ and the density-dependent diffusion coefficient D_M is defined as in the previous section. The constants $d_1, d_2 > 0$ and $k_1, \dots, k_5, \alpha, \beta \geq 0$. Moreover, we assume the initial data is smooth, satisfies the boundary conditions and

$$0 \leq C_0(x),\ A_0(x),\ M^0_0(x),\ M^1_0(x) \leq 1, \quad x \in \Omega.$$

As in the single species biofilm model, the existence of solutions is shown by a non-degenerate approximation of system (7.17). Under the assumption that the solutions satisfy L^∞-estimates, the positivity of the densities (of the auxiliary system) follows by Theorem 2.3. Sufficient for the boundedness of the solutions is the condition $\gamma \geq \alpha + \beta$.

Remark 7.2. *In a forthcoming paper we will consider the positive invariance of the positive cone for systems of stochastic partial differential equations (cf. [2]).*

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